Interface growth and Burgers turbulence: The problem of random initial conditions

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We study the relaxational dynamics of the deterministic Burgers equation, with random initial conditions, in an arbitrary spatial dimension d. In this paper we concentrate mainly on initial distributions relevant to interface growth rather than Burgers turbulence (although we shall present results for this system in $d=1$). By using an analytic approach, we are able to calculate both the short- and long-time forms for the kinetic energy of the fluid (or equivalently the roughness of the interface.) We find exponents describing the early-time behavior of the system.

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I. INTRODUCTION

The Burgers equation is one of the simplest models of fluid turbulence [1]. This equation may be derived from the Navier-Stokes equation by taking the fluid to be infinitely compressible and vorticity-free. The Burgers equation for the fluid velocity $\mathbf{v}(x,t)$ has the form

$$\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v} - \tfrac{1}{2} \nabla(\mathbf{v}^2) , \tag{1.1}$$

where ν is the viscosity. This equation is deterministic and describes the relaxation of the fluid from some (turbulent) initial condition $Q[\mathbf{v}(\mathbf{x},0)]$. The addition of a stochastic source to the right-hand side of (1.1) is appropriate to the problem of a randomly stirred fluid [2]. This equation has applications in many areas other than fluid turbulence due to the fact that it is one of the simplest nonlinear diffusion equations. The transformation $\mathbf{v} = -\lambda \nabla h$ (recall that $\nabla \times \mathbf{v} = 0$) produces the Kardar-Parisi-Zhang (KPZ) equation [3]

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 , \tag{1.2}$$

which is a popular model of interface growth. [The parameter λ that appears in (1.2) allows h to have dimensions of length appropriate to its interpretation as the interface height.] Again note that the above equation is deterministic and so describes the *relaxation* of an interface from some (disordered) initial condition. Interface growth in the context of ballistic deposition is described by (1.2) with the addition of a noise term. The stochastic versions of these equations have been extensively studied and are well understood in $d=1$. In higher dimensions there is little consensus on the values of the dynamic exponents related to the Burgers equation [3–8]. This is due to varying results from simulations (whose microscopic rules are assumed to be described by (1.2) in the long-wavelength regime [9]) and a lack of analytic results for $d \geq 2$.

In this paper we shall concentrate on the deterministic relaxation of these models. For purposes of calculation we shall exclusively use (1.2) (purely because it is simpler to deal with a scalar field). The most important result known for this problem is due to Burgers [1]–by exact analytic methods he showed that in $d=1$ there exists a scaling regime for the asymptotic ($t \to \infty, \nu \to 0$) behavior of the system. The scaling may be expressed in terms of a characteristic length scale $L(t) \sim t^{2/3}$. Dimensional analysis then indicates that the kinetic energy $E(t) = \frac{1}{2} \langle \mathbf{v}^2 \rangle_Q \sim t^{-2/3}$ (where the angled brackets indicate an average over the ensemble of initial conditions which in this case are Gaussian). Extension of his method to higher dimensions is very difficult. In the present work we shall present a new formulation of this problem which allows one to work in higher dimensions without too much extra difficulty.

The outline of the paper is as follows. In Sec. II we shall describe some simple steps which lead to a path-integral representation for $E(t)$ for a given distribution of initial conditions. An exact evaluation of the path integral is possible for an initial distribution which is decoupled in space:

$$P[h_0] = \prod_n p(h_0(x_n)) , \tag{1.3}$$

where h_0 is the initial value of the field h and the product is over all sites in the discretized space. In Sec. III we shall evaluate the exact form of the energy for both short and long times for a variety of such distributions. Clearly these results are relevant to interface relaxation [where $E(t) = \lambda^2 \langle (\nabla h)^2 \rangle_P / 2$ is a measure of the interface roughness], since the corresponding fluid distribution is ill defined. One of the most interesting results we shall derive is that for a distribution of the form (1.3) with p a bounded, continuous function, the short-time evolution of the energy has the form $E(t) \sim t^{-\sigma}$ where $\sigma = 2(d+1)/(d+2)$. In Sec. IV we concentrate on the case where $P[h_0] \sim \exp - \int (\nabla h_0)^2 d^d x$ which is relevant to the Burgers fluid and also to interfaces with smooth (continuous) initial profiles. We shall see that in this case, the path integral derived in Sec. II may now be interpreted as a field theory. The corresponding action is seen to be closely related to the Liouville model of string theory [10]. We evaluate the path integral in $d=1$ and confirm the main results of Burgers. Extensions of this method to higher dimensions are briefly discussed. Section V con-

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cludes this paper with a summary of our results and a discussion of future directions of research using this method.

II. PRELIMINARY STEPS

In the present work we shall calculate the kinetic-energy density of the fluid

$$E(t)\equiv\tfrac{1}{2}\langle \mathbf{v}^2\rangle_Q=\frac{\lambda^2}{2}\langle(\nabla h)^2\rangle_P\ , \qquad (2.1)$$

where $\langle\ \rangle_R$ indicates an average over the distribution R. We stress that the methods to be presented may be applied to other quantities of interest such as velocity-velocity correlation functions. This will be the subject of future work. We note from (1.2) and (2.1) that

$$E(t)=\lambda\partial_t\langle h(\mathbf{x},t)\rangle_P\ , \qquad (2.2)$$

where we have used translational invariance to drop the diffusion term.

An exact solution of (1.2) in terms of h_0 is possible by use of the Hopf-Cole transformation $h=(2\nu/\lambda)\ln w$ which leads to a simple diffusion equation for $w(\mathbf{x},t)$. We therefore have as the solution of (1.2)

$$h(\mathbf{x},t)=\alpha^{-1}\ln\int d^dy\, g(\mathbf{x}-\mathbf{y},t)\exp[\alpha h_0(\mathbf{y})]\ , \qquad (2.3)$$

where $\alpha=\lambda/2\nu$ and $g(\mathbf{x},t)=(4\pi\nu t)^{-d/2}\exp[-(x^2/4\nu t)]$ is the heat kernel.

It is clear from (2.2) and (2.3) that in order to evaluate the energy, we need to perform an average over the logarithm of a space integral. One possibility is to express the right-hand side of (2.3) as an arbitrary power of the space integral by use of the replica trick [11]. This is not a promising direction to take. We prefer to use the following representation of the logarithm function:

$$\ln z=\int_0^\infty\frac{du}{u}(e^{-u}-e^{-uz})\ . \qquad (2.4)$$

We now have to perform an average over the exponential of a space integral, which is a far more familiar task.

So combining (2.2)–(2.4) we have

$$E(t)=2\nu\partial_t\int_0^\infty\frac{du}{u}[e^{-u}-\psi(u,t)]\ , \qquad (2.5)$$

where

$$\psi(u,t)=\left\langle\exp-u\int d^dy\, g(\mathbf{y},t)e^{\alpha h_0(\mathbf{y})}\right\rangle_P\ . \qquad (2.6)$$

We now proceed with the evaluation of ψ for a given initial distribution P.

III. DISCRETE INITIAL DISTRIBUTIONS

In this section we shall investigate the form of $E(t)$ for discrete initial distributions—i.e., those which decouple in space as in (1.3). It is necessary in this case to bound the range of h_0. Let us consider the simplest possible case—a uniform distribution for which $|h_0|\le H$. Introducing a lattice cutoff l and performing the rescalings $u\to u=ul^de^K\tau^{-d/2}$, $y\to y=y(\tau/\pi)^{-1/2}$ (where $\tau=4\pi\nu t$ and $K=\alpha H$) we have, after evaluating the path integral,

$$E(t)=\frac{2\nu}{t}\left[\tau^{d/2}\int_0^\infty\frac{du}{u}\phi(u)\exp[-\tau^{d/2}\phi(u)]-1\right]\ , \qquad (3.1)$$

where

$$\phi(u)=-\pi^{-d/2}\int d^dy\,\ln\frac{1}{2K}\times[E_1(ue^{-y^2-2K})-E_1(ue^{-y^2})] \qquad (3.2)$$

and $E_1(z)$ is the exponential integral [12]. In (3.1) and (3.2) all lengths are scaled to have units of l.

The function $\phi(u)$ is quite rich. We may simplify it greatly by taking $K\gg1$. This corresponds physically to the strong turbulence limit. Using integration by parts and the series expansion for the exponential integral we find

$$2K\Gamma\left[\frac{d}{2}+1\right]\phi(u)=\int_0^\infty dy\, y^{d/2}(1-\exp-ue^{-y})+O(K^{-1})\ . \qquad (3.3)$$

We now rewrite (3.1) as

$$E(t)=\frac{2\nu}{t}\{F(\tilde{\tau})-1\}\ , \qquad (3.4)$$

where

$$F(\tilde{\tau})=\tilde{\tau}^{d/2}\int_0^\infty\frac{du}{u}\tilde{\phi}(u)\exp[-\tilde{\tau}^{d/2}\tilde{\phi}(u)]\ , \qquad (3.5)$$

with $\tilde{\phi}(u)=2K\phi(u)$ and

$$\tilde{\tau}=\tau(2K)^{-2/d}=4\pi\nu t\left[\frac{\nu}{\lambda H}\right]^{2/d}\ . \qquad (3.6)$$

We see that $\tilde{\tau}$ is the only relevant parameter and we shall now find the form of $E(t)$ in the limiting cases of $\tilde{\tau}\ll1$ and $\tilde{\tau}\gg1$.

It is clear from (3.5) that for $\tilde{\tau}\ll1$ the integral over u is dominated by large u. We therefore need the asymptotic form of $\tilde{\phi}(u)$ for $u\gg1$. We find from (3.3) that for $u\gg1$

$$\tilde{\phi}(u)=\frac{(\ln u)^{d/2+1}}{\Gamma(d/2+2)}+\gamma\frac{(\ln u)^{d/2}}{\Gamma(d/2+1)}+O((\ln u)^{d/2-1})\ , \qquad (3.7)$$

where γ is Euler's constant. Using this asymptotic form for $\tilde{\phi}(u)$ we find from (3.5)

$$F(\tilde{\tau})=\frac{2}{(d+2)}\Gamma\left[\frac{d+4}{d+2}\right]\left[\frac{\Gamma[(d+4)/2]}{\tilde{\tau}^{d/2}}\right]^{2/(d+2)}+O(1)\ . \qquad (3.8)$$

Inserting (3.8) into (3.4) we have the following result for the energy:

$$E(t)=\frac{4\lambda^2}{(d+2)}\Gamma\left[\frac{d+4}{d+2}\right]\times\left[\frac{H\Gamma[(d+4)/2]}{(4\pi)^{d/2}(\lambda t)^{(d+1)}}\right]^{2/(d+2)}+O(t^{-1}) . \quad (3.9)$$

So for "short times" (in the strong-turbulence limit the crossover time may in fact be very large) such that $t \ll (4\pi\nu)^{-1}(\lambda H/\nu)^{2/d}$ we find that the energy decays as $E(t)\sim t^{-\sigma}$ where $\sigma=2(d+1)/(d+2)$.

The case of $\tilde{\tau}\gg 1$ is much simpler to analyze. Here we need the small-u form of $\tilde{\phi}$ which is easily found to be

$$\tilde{\phi}=u-2^{-(d+4)/2}u^2+O(u^3) . \quad (3.10)$$

Inserting this expansion into (3.5) we have

$$F(\tilde{\tau})=1+\tfrac{1}{4}(2\tilde{\tau})^{-d/2}+O(\tilde{\tau}^{-d}) . \quad (3.11)$$

Therefore using (3.11) and (3.4) we find

$$E(t)=\frac{H\lambda}{2(8\pi\nu)^{d/2}t^{d/2+1}}[1+O(t^{-d/2})] \quad (3.12)$$

for $\tilde{\tau}\gg 1$. This result corresponds to simple diffusion.

We expect the above results (i.e., the exponents for the energy decay) to be valid for any initial distribution of the form (1.3) so long as the single-site distribution function is continuous in h, and bounded. This may be verified explicitly for the case of a bounded Poisson distribution. An interesting alternative is to consider a discrete single-site distribution function. We shall briefly discuss the case of

$$p(h_0)=\tfrac{1}{2}\delta(h_0+H)+\tfrac{1}{2}\delta(h_0-H) , \quad (3.13)$$

i.e., a bimodal distribution.

Following the familiar steps the energy may be expressed as in (3.1) with

$$\Gamma\left[\frac{d}{2}+1\right]\phi(u)=u\int_0^\infty dy\, y^{d/2}e^{-y}\left\{\frac{\exp-ue^{-y}}{1+\exp-ue^{-y}}\right\}+O(e^{-2K}) . \quad (3.14)$$

The relevant parameter is now simply $\tau=4\pi\nu t$. For $\tau\ll 1$ we again need the large-u form of ϕ which is found to be

$$\phi(u)=\frac{\ln 2[\ln(u)]^{d/2}}{\Gamma(d/2+1)}[1+O((\ln u)^{-1})] . \quad (3.15)$$

We then obtain from (3.1) and (3.15)

$$E(t)=\frac{\Gamma[(d+2)/d]}{\pi d}\left[\frac{\Gamma(d/2+1)}{\ln 2}\right]^{2/d}t^{-2}+O(t^{-1}) \quad (3.16)$$

for $\tau\ll 1$. This result of $E(t)\sim t^{-2}$ independent of d may be understood by considering that the early-time fluctuations in the bimodal case relax independently [13] (in the language of Burgers turbulence, there is no coalescence of shock waves in this case). Interaction processes only occur for initial fluctuations which have a continuous range of sizes as in the first example studied in this section.

For the case of $\tau\gg 1$ we obtain

$$E(t)=\frac{2\nu}{(8\pi\nu)^{d/2}t^{d/2+1}}[1+O(t^{-d/2})] , \quad (3.17)$$

which is again the result corresponding to late-time diffusion.

As a final comment in this section we point out that an alternative way of calculating $E(t)$ is to evaluate the two-point correlation function

$$C(\mathbf{r},t)=\langle [h(\mathbf{r},t)-h(0,t)]^2\rangle_P \quad (3.18)$$

and then to take the limit

$$E(t)=\frac{\lambda^2}{2}\lim_{r\to 0}\frac{C(\mathbf{r},t)}{r^2} . \quad (3.19)$$

This calculation has been performed for the case of the uniform initial condition and the results are the same as those obtained above. [In order to calculate $C(\mathbf{r},t)$ one simply applies the logarithm representation (2.4) twice and then proceeds similarly to the above analysis.] Of course $C(\mathbf{r},t)$ is very interesting in its own right. One would like to calculate it explicitly in order to determine the existence of scaling and so forth. So far we have only been able to calculate the limit expressed in (3.19).

IV. CONTINUOUS DISTRIBUTIONS

Continuous distributions are more relevant to the study of Burgers turbulence. An interesting case to study is

$$P[h_0]\sim\exp[-(1/4D)\int(\nabla h_0)^2 d^dx] , \quad (4.1)$$

which corresponds to a simple Gaussian initial condition for the Burgers fluid. This distribution function is also relevant to strongly disordered, but smooth interfaces. Referring to (2.5) we see that the same expression holds for the energy, where now

$$\psi(u,t)=\int\mathcal{D}h_0\exp(-S[h_0;u,t]) . \quad (4.2)$$

This is written in the form of a field theory defined by the following action:

$$S[h_0;u,t]=\int d^dx\left\{\frac{1}{4D}(\nabla h_0)^2+ug(\mathbf{x},t)e^{\alpha h_0(\mathbf{x})}\right\} . \quad (4.3)$$

This is closely related to the Liouville model which has received some attention in string theory [10], but differs by the existence of the symmetry breaking heat kernel that multiplies the "potential" term. The study of this action using field-theory methods (saddle-point approximation, renormalization group) is an interesting topic for future research. In the present work, we shall limit ourselves to studying the case $d=1$, where the problem may be recast in the form of a Schrödinger equation, the solution of which will lead us to obtain the asymptotic form of the energy $E(t)\sim t^{-2/3}$ in agreement with the earlier work of Burgers. Before continuing we note that the above field theory has an upper critical dimension of $d=2$ which implies that the asymptotic ($t\to\infty$) evolution of the interface for $d>2$ is diffusive, $E(t)\sim t^{-d/2}$ (note that the power law for diffusive behavior differs from that in Sec. III due to the smooth initial condition). This may be verified explicitly by naive perturbation theory for (4.3)—there are no divergences for $d>2$.

Consider now the one-dimensional case (quantum mechanics). We denote as $\Phi(h,y)$ the density of the function ψ along the h axis—it is a sum over all the paths terminating at the point h for a given "time" y. This function obeys the forward and backward diffusion equation with a decay term

$$\frac{\partial\Phi(h,y)}{\partial y}=\pm D\frac{\partial^2\Phi}{\partial h^2}\mp ug(y)e^{\alpha h}\Phi\ , \tag{4.4}$$

and is subject to the following condition. Since the solution (2.3) is translationally invariant in all directions including h itself, we may choose the initial condition in the path integral (4.2) to be $h_0(0)=0$, which results in $\Phi(h,0)=\delta(h)$. The "forward" and "backward" forms of Eq. (4.4) are then considered on the right and left half axis of y, respectively, and have identical solutions. In terms of the right half-axis solution, $\Phi(h,y)$, we have

$$\alpha\langle h\rangle=\int_0^\infty\frac{du}{u}\left[e^{-u}-\left[\int_{-\infty}^{\infty}\Phi(h,\infty)dh\right]^2\right]\ . \tag{4.5}$$

Expansion of Eq. (4.4) in powers of u shows that the actual parameter of such an expansion is $u\exp(\nu\alpha^4D^2t)$, and so the representation values of $u\sim\exp[-\alpha D^{2/3}(\lambda t)^{1/3}]$, to be obtained shortly, belong to the nonperturbative region. The exponential entering the decay term, namely, $\exp(-y^2/4\nu t+\alpha h)$, suggests that there is a line in the (h,y) plane

$$h^*(y)=h_0+y^{2/}(2\lambda t),\quad h_0=-\alpha^{-1}\ln u\ , \tag{4.6}$$

where decay is not negligible. We note that the approximation used by Burgers [1] can be understood on the basis of Eq. (4.4) if one drops the decay term in the equation and introduces instead zero boundary condition on the curve, $\Phi(h^*(y),y)=0$. It is clear at this stage that the numerical prefactors of the various averaged quantities given in Ref. [1] *are affected* by such an approximation.

The qualitative effect of the decay term in (4.4) may be seen from a numerical solution of this equation. The function $\ln\Phi$ is displayed in Figs. 1(a) and 1(b) for the cases of $u=0$ and 0.1, respectively. We see that the effect of the decay term is to erode the function for large enough $h>0$. Semiquantitative investigation shows that the net decay is not negligible if the support of the diffusion Green function of (4.4), $\sqrt{Dy}$, intersects the curve $h^*(y)$. These two curves touch each other if $h_0\sim D^{2/3}(\lambda t)^{1/3}$—this happens at the point $(h,y)\sim(D^{2/3}(\lambda t)^{1/3},D^{1/3}(\lambda t)^{2/3})$, where we have omitted numerical prefactors.

The representation (2.4) allows us to obtain results with so-called logarithmic precision. Indeed, if ψ is unity at $u\ll u_0$ and otherwise exponentially small, the u integral is just $-\ln(u_0)=\alpha h_0$. This results in the kinetic-energy decay

$$E(t)=\eta\left[\frac{D\lambda^2}{t}\right]^{2/3}\ , \tag{4.7}$$

which is essentially the Burgers result. The prefactor η can be obtained by solving Eq. (4.4) numerically. In Fig. 2 is presented the time dependence of $\alpha\langle h\rangle$ (which we expect to behave as $t^{1/3}$). The y intersect of this graph is equal to $\ln(3\eta/2)$ whence we find $\eta=0.499(1)$.

V. SUMMARY AND CONCLUSIONS

We have presented an analytic study of the deterministic Burgers equation with random initial conditions. This has been made possible by the use of (2.4) which enables us to write quantities of interest (in this case, the energy) as simple path integrals. In Sec. III, we explicitly solved the case of a discrete initial distribution which is relevant to interfaces which relax from a strongly disordered, discontinuous initial form. In general there are two regimes in the evolution, the later one being simple

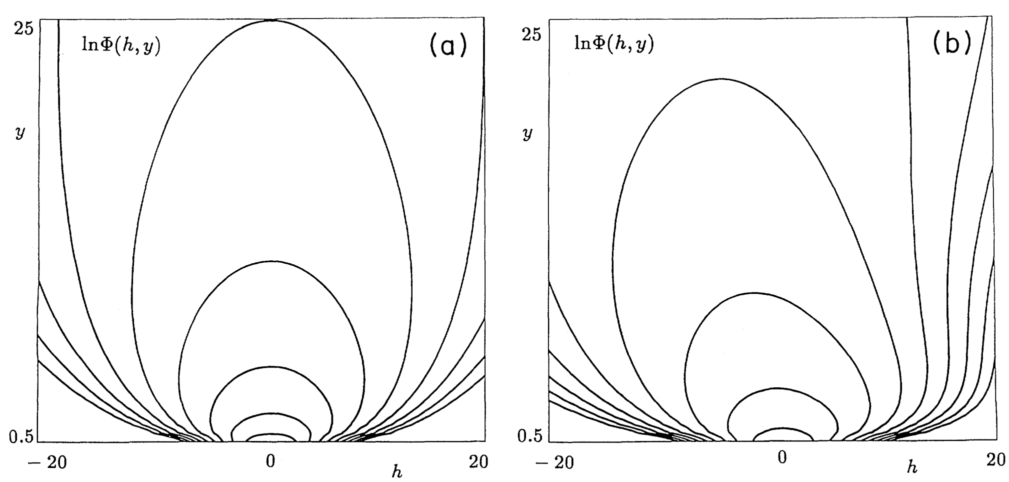

FIG. 1. Contour plot of $\ln\Phi$ for (a) $u=0$ and (b) $u=0.1$. The parameters in (4.4) have values $t=40$, $\nu=1$, $\lambda=1$, $D=1$.

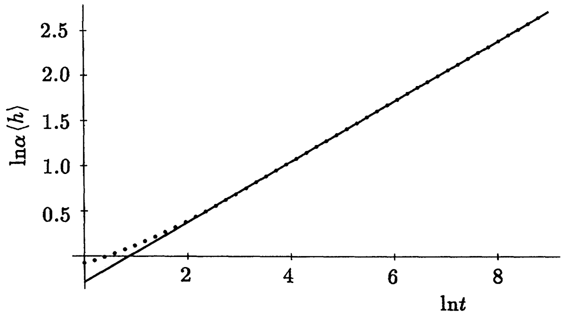

FIG. 2. Log-log plot of $\alpha\langle h\rangle$ vs t. The slope is 0.3334(1).

diffusion. For the case of a uniform single-site distribution, we found a new, nontrivial exponent for the energy (roughness) decay in the early-time regime: $E(t)\sim t^{-\sigma}$, where $\sigma=2(d+1)/(d+2)$. For the bimodal initial condition we found the d-independent exponent $\sigma=2$. Forms for the crossover times were calculated explicitly.

In Sec. IV we studied the case of a continuous initial profile, which corresponds to a simple Gaussian initial condition for the Burgers fluid. In this case, the use of (2.4) leads one to a natural field-theory representation for the energy (and velocity-velocity correlation functions) which closely resembles the Liouville model of string theory. We solved the case $d=1$ by using the correspondence of the path integral to a quantum-mechanical transition amplitude, and recovered the result of Burgers: $E(t)\sim t^{-2/3}$. The asymptotic form of the energy for $d>2$ was noted to be diffusive due to the upper critical dimension of the field theory being $d=2$.

There are several interesting directions to explore using the present description of Burgers turbulence. First one may study (4.3) using standard field-theory techniques in order to have a more complete understanding of the energy decay (crossover times etc.) in arbitrary spatial dimensions. Also one should be able to calculate correlation functions using similar steps to those presented here. This is crucial for a complete understanding of scaling behavior in these systems. It would be nice to obtain more exact results from this method, since then the Burgers equation could serve as a testing ground in the field of nonequilibrium systems for the application of other methods such as mode-coupling theories and the dynamic renormalization group.

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